KVS Junior Mathematics Olympiad (JMO)
SAMPLE PAPER – 7

M.M. 100 Time : 3 hours

Note : Attempt all questions.
All questions carry equal marks

Q1. Simplify : $\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1}$

Q.2 For the set of questions
$z^x = y^{2x},\ 2^z = 2.4^x,\ x+y+z = 16$
find the integral values of x, y, z

Q.3 Solve the equation : $x^2 - 2|x| - 3 = 0$

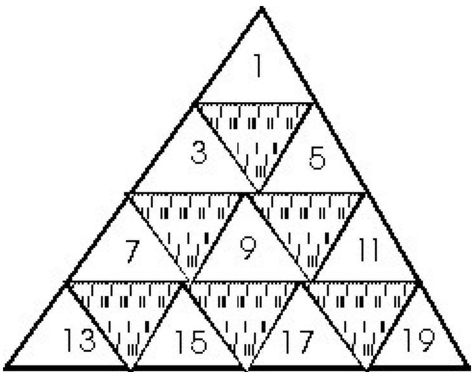
Q.4 Prove that in any triangle, the sum of medians is more than $\frac{3}{4}$ of its perimeter, but less than the whole perimeter.

Q. 5 The right triangle is having sides a,b and c. Find the radius of the inscribed circle in terms a, b and c.

Q.6 Transpose the letter with digits
ACID +BASE = SALT + H2O
In H2O, the "O" is a digit, not a letter.

Q.7 Two tins containing originally 20 litres of milk and 10 liters of water respectively. Four litres of the liquid are not drawn from each tin and placed in the other, the liquid being thoroughly mixed. The same process is repeated a second time. Find the ratio of milk and water in each tin after the final mixture.

Q. 8 The following pattern continues and numbers in the 100th row is added, find the value.



Q. 9 Determine which of the two numbers $(1000)^{1000}$ and $(1001)^{999}$ is greater ?

Q.10 Find a four digit number which, on division by 131 yields a remainder of 112 and on division by 132 yields a remainder of 98.

Q.1 $\dfrac{1}{X-1}-\dfrac{1}{X+1}-\dfrac{2}{X^2+1}-\dfrac{4}{X^4+1}$

$= \dfrac{X+1-X+1}{X^2-1}-\dfrac{2}{X^2+1}-\dfrac{4}{X^4+1}$

$= \dfrac{2}{X^2-1}-\dfrac{2}{X^2+1}-\dfrac{4}{X^4+1}$

$= \dfrac{2\left[X^2+1-X^2+1\right]}{X^4-1}-\dfrac{4}{X^4+1}$

$= \dfrac{4}{X^4-1}-\dfrac{4}{X^4+1}$

$= \dfrac{4\left[X^4+1-X^4+1\right]}{X^8-1}$

$= \dfrac{8}{X^8-1}$

Q.2 $Z^x = Y^{2x}$ (i)

$2^z = 2\cdot 4^x$ (ii)

$x + y + z = 16$ (iii)

from (i) $\Rightarrow$ $z^x = \left(y^2\right)^x \Rightarrow z = y^2$ (iv)

(ii) $\Rightarrow$ $2^z = 2^{1+2x} \Rightarrow z = 1+2x$ (v)

from (iv) and (v) $\Rightarrow$ $y^2 = 1+2x \Rightarrow x = \dfrac{y^2-1}{2}$ (vi)

$\therefore$ (iii), (iv) (vi) $\Rightarrow \dfrac{y^2-1}{2}+y+y^2 = 16$

$\Rightarrow$ $y^2-1+2y+2y^2 = 32$

$\Rightarrow$ $3y^2 + 2y - 33 = 0$

$\Rightarrow$ $y\,(3y+11) - 3\,(3y+11) = 0$

$\Rightarrow$ y = 3, $y = \dfrac{-11}{3}$

But they are integers.

So y = 3

$\Rightarrow$ x = 4 and z = 9

Q.3 $x^2 - 2|x| - 3 = 0$

For $x \geq 0$ $|x| = x$

So $x^2 - 2x - 3 = 0$

$\Rightarrow$ (x-3) (x+1) = 0

$\Rightarrow$ x=3 or x = -1

But $x \geq 0$ so x=-1 is not possible

$\therefore$ x = 3

Again for $x < 0$ $|x| = -x$

So $x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3 \; or \; x = -1$

but $x \geq 0$ so x = -1 is not possible

$\therefore$x = 3

Again for x<0 $|x| = -x$

So $x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0$

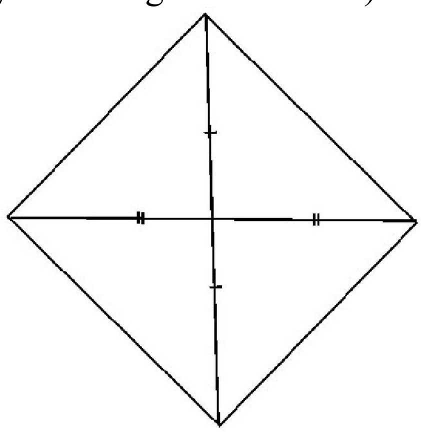
$\therefore$ x = -3 or x = 1

But x =1 in not possible as x<0

So x = -3

So x = ±3

Q.4 In arbitrary triangle ABC is the median through ∠C. In Δ CBE and Δ AED (on formation of other triangle) by extending median to D.)



Two sides and a between angle is equal. So the Δ CBE and Δ AED are congruent. So AD=CB from side inequality

AD + AC > CD

$\Rightarrow$ AC + BC > 2CE

if sides are a, b, c then medians are m_a, m_b and m_c.

So $b + a > 2m_c$

Similarly $a + c > 2m_b$ and $c + b > 2m_a$

Adding we get $a + b + c > m_a + m_b + m_c$

So sum of medians is less than whole perimeter.

For CE = m_c

$CM = \frac{2}{3} m_c$

Here AM + MC > AC

Similarly, $\frac{2}{3} m_c + \frac{2}{3} m_a > b$

$\frac{2}{3} m_a + \frac{2}{3} m_b > c$

and $\frac{2}{3} m_b + \frac{2}{3} m_c > a$

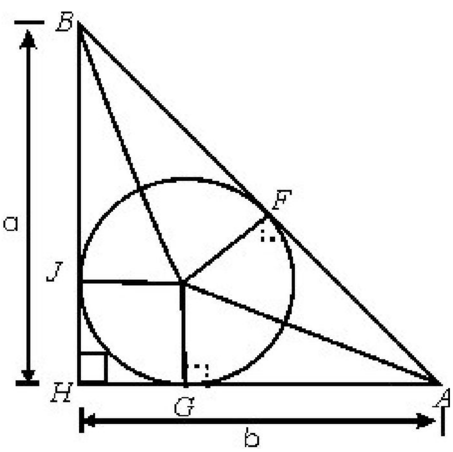
Adding we get

$m_a + m_b + m_c > \frac{3}{4}(a+b+c)$

so, sum of medians is more than $\frac{3}{4}$ of its perimeter.

Q5. In right angled triangle AHB, $\angle H = 90^o$ AO is bisector of $\angle A$ BO is bisector of $\angle B$ OF = OG

So ΔOFA and Δ OGA are congruent.



So, AF = AG

= b – r

BF = BJ

= a – r

so, AB = (a + b) – 2r

$\Rightarrow 2r = (a+b) - AB \quad \Rightarrow r = \frac{(a+b) - \sqrt{a^2 + b^2}}{2}$

$= \frac{a+b-c}{2}$

$c^2 = a^2 + b^2$

Q.6 Solution is :

A = 6	H = 1
A = 2	I = 7
C = 0	L = 3
D = 5	S = 8
E = 4	T = 9

Q. 7.

After Operation 1

	Tin 1		Tin 2	
	Milk	Water	Milk	Water
After Operation 1	16	4	4	6
Contents In 4 litres of mixture taken our for operation 2	4 X 16/20 = 16/5	4X4/20 = 4/5	4 X 4/10 = 16/10	4x6/10 = 24/10
Remaining Just after taking out 4 litres of mixture	16-16/5	4-4/5	4-16/10	6-24/10
After mixing (adding) the 4 litres of mixtures	16-16/5 + 16/10	4-4/5 + 24/10	4-16/10 + 16/5	6-24/10 + 4/5
So ratio milk / water	144/10 ÷56/10 = 18/7 i.e. 18:7		56/10 ÷44/10 = 14:11	

Q. 8 In the pattern

$=1=1^3$

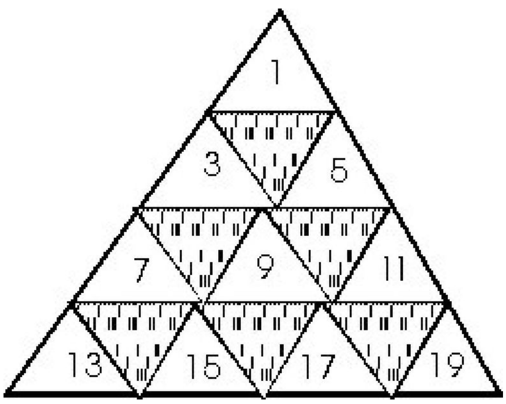
$=8=2^3$

$=27=3^3$

$= 64=4^3$

So 100^{th} = 100^3

= 1 00 00 00



Q. 9 Let us prove that $2 \le \left(1+\frac{1}{n}\right)^n \le 3$

Here $1+\frac{K}{n} \le \left(1+\frac{1}{n}\right)^k < 1+\frac{k}{n}+\frac{k^2}{n^2}$

For any positive integer k such that k ≤ n.
By mathematical induction
For K = 1

$$1+\frac{1}{n} = 1+\frac{1}{\text{n}} \le 1+\frac{1}{n}+\frac{1}{n^2}$$

for K+1

$$\left(1+\frac{1}{n}\right)^{k+1} = \left(1+\frac{1}{n}\right)^k \left(1+\frac{1}{n}\right) \ge \left(1+\frac{k}{n}\right)\left(1+\frac{1}{n}\right)$$

$$= 1+\frac{k+1}{n}+\frac{k^2}{n^2} > 1+\frac{k+1}{n}$$

Now for K ≤ n

$$\left(1+\frac{1}{n}\right)^{k+1} = \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right) < \left(1+\frac{k}{n}+\frac{k^2}{n^2}\right)\left(1+\frac{1}{n}\right)$$

$$= 1+\frac{k+1}{n}+\frac{k^2+2k+1}{n^2}-\frac{k+1}{n^2}+\frac{k^2}{n^3}$$

$$= 1+\frac{k+1}{n}+\frac{(k+1)^2}{n}-\frac{n(k+1)-k^2}{n^3}$$

$$< 1+\frac{k+1}{n}+\frac{(k+1)^2}{n^2}$$

because n(k+1) $\ge$ k^2
for k = n
and for n $\ge$ k
we obtain

$$2 = 1+\frac{n}{n} \le \left(1+\frac{1}{n}\right)^n \le 1+\frac{n}{n}+\frac{n^2}{n^2} = 3$$

so $2 \le \frac{(1+1)^{\text{n}}}{\text{n}} < 3$

for the current problem

$$\frac{(1001)^{999}}{(1000)^{1000}} = \left(\frac{1001}{1000}\right)^{1000} \cdot \frac{1}{1001}$$

$$= \left(1+\frac{1}{1000}\right)^{1000} \cdot \frac{1}{1001}$$

$$< 3 + \frac{1}{1000} < 1$$

so $(1000)^{1000} > (1001)^{999}$

Q.10 Say N is number

$N = 131k = 112$ k and l are positive

$N = 132l + 98$ integers N is four digit number

$$l = \frac{N-98}{132} < \frac{10000-98}{132} < 75.02$$

i.e. $l \leq 75$; further

$131k + 112 = 132l + 98$

$\Rightarrow 131 (k-1) = (l-14)$

$k - 1 \neq 0$ then l-14 exceeds 130 in absolute value, which is not getting possible if $l < 75$

so necessarily $k - 1 = 0$

$\Rightarrow k = 1$

yielding $k = 1 = 14$

$N = 131.14 + 112$

$= 132.14 + 98$

$= 1946$

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