KVS Junior Mathematics Olympiad (JMO)
SAMPLE PAPER – 2

M.M. 100 Time : 3 hours

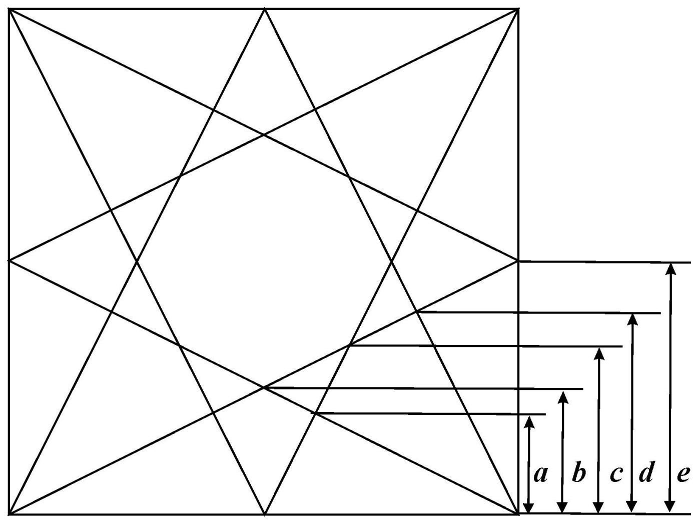
Note : Attempt all questions.

All questions carry equal marks

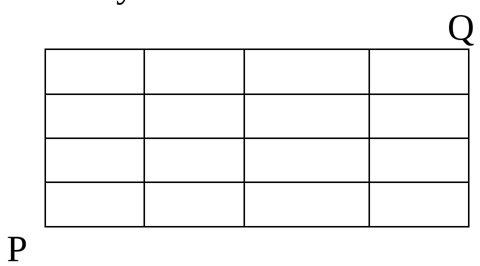
Q.1 Factorize

$$(y-z)^5 + (z-x)^5 + (x-y)^5$$

Q.2 Calculate the lengths a, b, c, d and e in terms of side of square. Take the figure symmetrical.



Q.3 In the square 4 x 4 as below, the movement allowed from P to Q only along black lines. How many different shortest routes are there from P to Q ?



Q.4 Prove that :

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+.........+\frac{2}{n(n+1)}<2$$

Q.5 Find the smallest integer K which when divided by 6, 5, 4, 3, 2 successively leaves remainders 5, 4, 3, 2 and 1 respectively.

Q.6 A cube is inscribed in a sphere. If the surface area of the cube is 60 cm^2, find the surface area of the sphere.

Q.7 Prove $11^{10}-1$ is divisible by 100.

Q.8 ABC is a right angled triangle, right angle at B. Let E be the mid point of side AC. A circle is drawn taking E as center such that the circle touches the side AB at D. If the length of side AC and AB are roots of quadratic equation $x^2 - px + q = 0$. Find the radius of the circle.

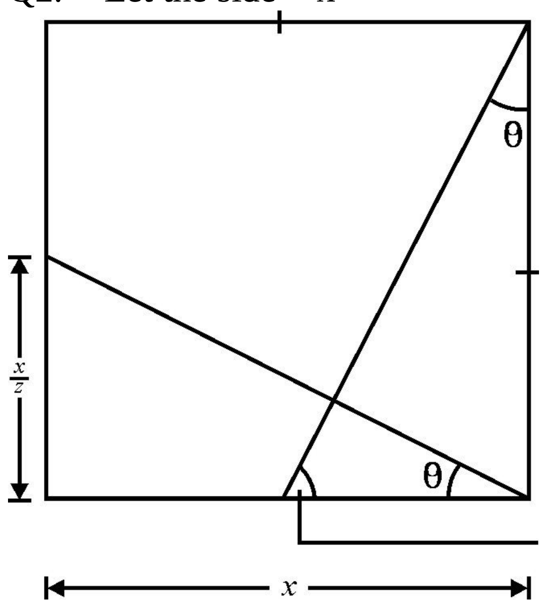
Q.9 How many squares and rectangles are there on standard chess board ?

Q.10 Mr. John was x years old in year x^2. When was John born if he died in 1871.

SOLUTIONS AND HINTS (SAMPLE PAPER 2)

Q1. Putting a, b, c for y-z, z-x and x-y respectively.
The expression reduces to
$a^5 + b^5 + c^5$
Now a + b + c = y-z + z – x + x – y = 0
$\Rightarrow a + b = - c$
$\Rightarrow a^2 + b^2 + 2ab = c^2$
$\Rightarrow - c^5 = (a + b)^5$
$= a^5 + b^5 + 5a^4b + 10a^3b^2 + 10a^2b^2 + 5ab^4$
$\Rightarrow - (a^5 + b^5 + c^5) = 5ab (a^3 + b^3) + 10a^2b^2 (a+b)$
$= 5ab (a + b) \{(a^2 – ab + b^2) + 2ab\}$
$= 5ab (-c) \{a^2 + b^2 + ab)$
$= - 5abc (a^2 + b^2 + ab)$
putting back the values of a, b and the expression
$= 5 (y-z) (z-x) (x-y) \{(y-z)^2 + (z-x)^2 + (y-z) (z-x)\}$
$= 5 (y-z) (z-x) (x-y) (y^2+z^2 -2yz + z^2–2zx+yz+2x-x^2 – xy)$
$(y-z)^5 + (z-x)^5 + (x-y)^5 = 5 (y-z) (z-x) (x-y) (x^2+ y^2+z^2 -yz -zx – xy)$

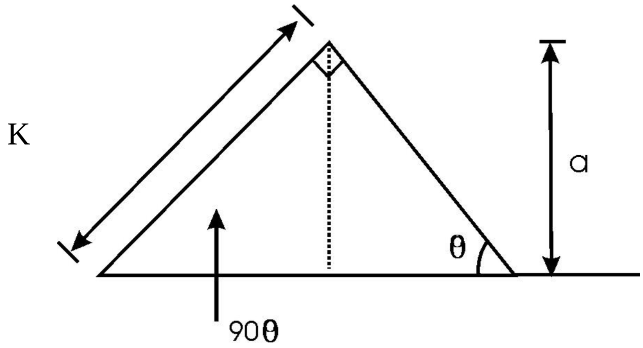
Q2. Let the side = x



This angle = 90-θ, so the point of intersection of two lines = 90^o

So for line = $\sqrt{x^2 + \frac{x^2}{4}}$

$= \sqrt{\frac{5}{2}} x$



in the Figure

$\text{Sin}\ \theta = \dfrac{K}{x/2}$

$\Rightarrow \dfrac{1}{\sqrt{5}} = \dfrac{K}{x/2}$

$\Rightarrow K = \dfrac{a}{2\sqrt{5}}$

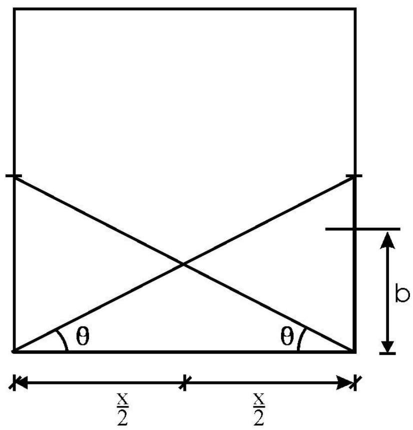
Further, $\sin(90 - \theta) = \dfrac{a}{K}$

$= \dfrac{a}{\frac{x}{2\sqrt{5}}}$

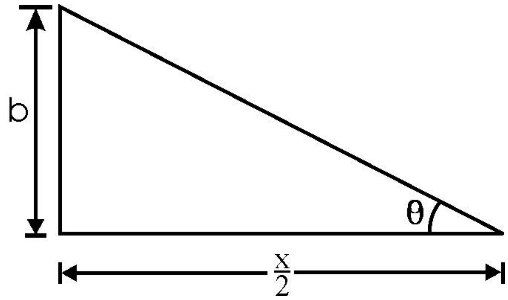
$\Rightarrow \dfrac{2}{\sqrt{5}} = \dfrac{a}{1} \times \dfrac{2\sqrt{5}}{x}$

$\Rightarrow a = \dfrac{x}{5}$ (i)

For figure



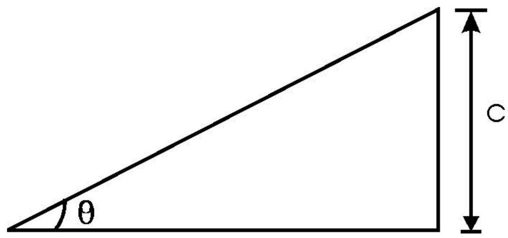
As two angles are equal, hence this is a Isoceseles



$$\text{Tan}\ \theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \frac{1}{2}$$

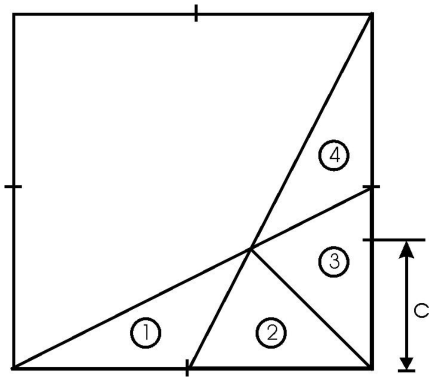
$$\Rightarrow \frac{1}{2} = \frac{b}{\frac{x}{2}}$$

$$\Rightarrow b = \frac{x}{4} \qquad \text{(ii)}$$



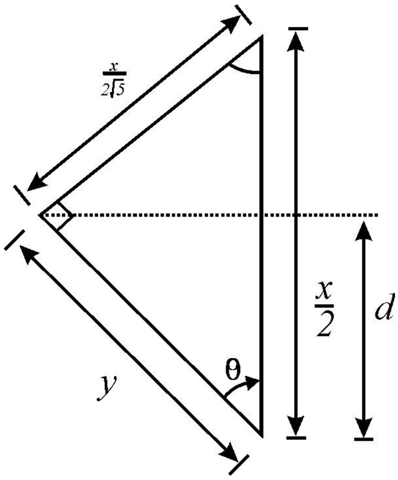
For c : let's divide
area Δ 1 = area Δ 2
area Δ3 = area Δ4

area (Δ 1+ Δ 2+ Δ 3) = $\frac{1}{2}.x.\frac{x}{2}$



$$\Rightarrow 3\,\Delta = \frac{x^2}{4}$$

$$\therefore\ \Delta = \frac{x^2}{12}\ \text{so c} = \frac{x^2}{12}\ x\ \frac{2x2}{x} = \frac{x}{3}\ \text{(iii)}$$



From the figure

$$\left(\frac{x}{2}\right)^2 = \left(\frac{x}{2\sqrt{5}}\right)^2 + y^2$$

$$\Rightarrow y = \frac{x}{\sqrt{5}}$$

$$\cos\theta = \frac{d}{y}$$

$$\Rightarrow \frac{d}{y} = \frac{2}{\sqrt{5}}$$

$$\Rightarrow d = \frac{2}{\sqrt{5}}\cdot\frac{x}{\sqrt{5}}$$

$$= \frac{2x}{5} \qquad \text{(iv)}$$

$$e = \frac{x}{2}$$

So answer is

$$a = \frac{x}{5}$$

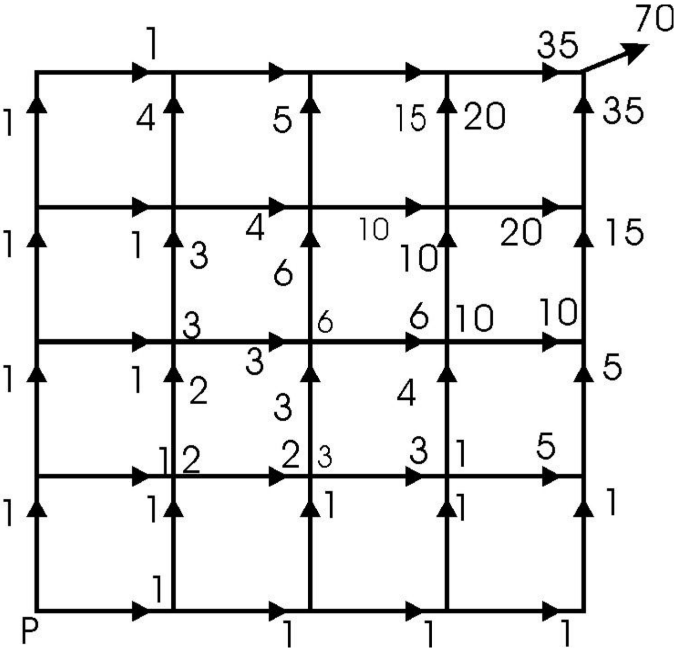
$$b = \frac{x}{4}$$

$$c = \frac{x}{3}$$

$$d = \frac{2}{5}x$$

$$e = \frac{x}{2}$$

Q3.



Let us make use of
Principle of PASCAL triangle
to total number of ways = 70 ways

Q.4 $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+.....+\frac{2}{n(n+1)}<2$

Here

$1=\frac{1x2}{2}$, $2=\frac{2x3}{2}$, $6=\frac{3x4}{2}$, $10=\frac{4x5}{2}$ and so on.

So,

$\frac{2}{1x2}=2\left(\frac{1}{1}-\frac{1}{2}\right)$

$\frac{2}{2x3}=2\left(\frac{1}{2}-\frac{1}{3}\right)$

$\frac{2}{3x4}=2\left(\frac{1}{3}-\frac{1}{4}\right)$

$\frac{2}{4x5}=2\left(\frac{1}{4}-\frac{1}{5}\right)$

$\frac{2}{5x6}=2\left(\frac{1}{5}-\frac{1}{6}\right)$

$\frac{2}{nx(n+1)}=2\left(\frac{1}{n}-\frac{1}{n+1}\right)$

on adding LHS = $2\left(\frac{1}{n}-\frac{1}{1+n}\right)$

$= 2 \times \frac{n}{n+1}$

As $\frac{n}{n+1}$ as always less than 1

So LHS < 2

Q5. LCM of 6, 5, 4, 3, 2, 1 = 60

So Number is 60 – 1 = 59

Q6. Surface area of cube = $6a^2$ (a is length of a side)

From Q; $6a^2 = 60$

$\Rightarrow$ $a^2 = 10$

The diameter of the sphere is the diagonal of the cube.

$\Rightarrow$ $(\text{diameter})^2 = a^2 + (\sqrt{2}\ a)2$

$= 3a^2$

so diameter = $\sqrt{3}\ a$

so surface area of sphere = $4\pi\ (\text{radius})^2$

$= 4\pi\ (\frac{\sqrt{3}}{2}\ a)^2$

$= 3\pi\ a^2$

$= 30\ \pi\ \text{cm}^2$

Q7. We know that all powers of 11 ends with 1.

So, 11k has the digit K in last but one position,

For k = 1, 2, 3 ….9.

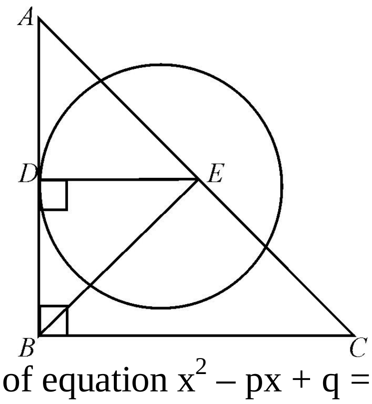
For 11^{10} has 0 in last but one position

2nd proof :

$11^{10} - 1 = (11-1)\ (11^9 + 11^8 + 11^7 + 11^6 + \ldots\ldots + 11^2 + 1)$

= 10 x 10 (As unit digit of ten numbers are 1 each)

Q8.



Given AB an AC are the roots of equation $x^2 - px + q = 0$

$\therefore$ AB + AC = p

AB.AC = q

$\therefore$ AB - AC = $\pm\ \sqrt{p^2 - 4q}$

$\therefore\ AB = \dfrac{p \pm \sqrt{p^2 - 4q}}{2}$

$AC = \dfrac{p \pm \sqrt{p^2 - 4q}}{2}$

$AB > AC$

So $\quad AC = \dfrac{p + \sqrt{p^2 - 4q}}{2}$

$\quad AB = \dfrac{p - \sqrt{p^2 - 4q}}{2}$

$AC^2 = AB^2 + BC^2$

$$\mathbf{BC^2 =} \left(\frac{p + \sqrt{p^2 - 4q}}{2}\right)^2 - \left(\frac{p - \sqrt{p^2 - 4q}}{2}\right)^2$$

$= \sqrt{p\sqrt{p^2 - 4q}}$

Now DE || BC

$\therefore\ \Delta ADE \sim \Delta ABC$

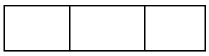
$\therefore\ \dfrac{AE}{DE} = \dfrac{AC}{BC}$

$\Rightarrow\ DE = \dfrac{BC}{AC} x AE \ = BC x \dfrac{AE}{2AE} = \dfrac{1}{2} BC$

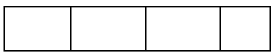
$= \dfrac{1}{2} x \sqrt{p\sqrt{p^2 - 49}}$

Q9. 1 x 2 (2 unit) rectangles = 7 x 8 + 7 x 8

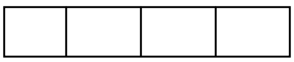
1 x 3 (3 unit) rectangles = 6 x 8 + 6 x 8



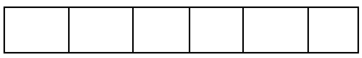
1 x 4 (4 unit) rectangles = 5 x 8 + 5 x 8



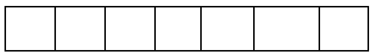
1 x 5 (5 unit) rectangles = 4 x 8 + 4 x 8



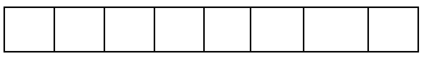
1 x 6 (6 unit) rectangles = 3 x 8 + 3 x 8



1 x 7 (7 unit) rectangles = 2 x 8 + 2 x 8



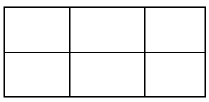
1 x 8 (8 unit) rectangles = 1 x 8 + 1 x 8



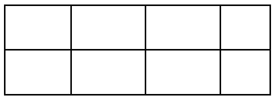
(1+2+3+4+5+6+7) x 8 x 2 = 448

Similarly,

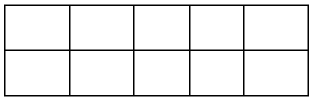
(2 x 3) i.e. 6 unit rectangles = 6 x 7 + 6 x 7



(2 x 4) i.e. 8 unit rectangles = 5 x 7 + 5 x 7



(2 x 5) i.e. 10 unit rectangles = 4 x 7 + 4 x 7



(2 x 6) i.e. (12 unit) rectangles = 3 x 7 + 3 x 7

(2 x 7) i.e. (14 unit) rectangles = 2 x 7 + 2 x 7

(2 x 8) i.e. (16 unit) rectangles = 1 x 7 + 1 x 7

total = 2 x 21 x 7 = 294

Similarly for

3 x 4 to 3 x 8 rectangles	=	180
4 x 5 to 4 x 8 rectangles	=	100
5 x 6 to 5 x 8 rectangles	=	48
6 x 7 to 6 x 8 rectangles	=	18
7 x 8 rectangles	=	4

totaling 4 + 18 + 48 + 100 + 180 + 294 + 448 = 1092

Further there are

$1^2 + 2^2 + 3^2 + 4^2 + \ldots + 8^2$ squares

of 8 x 8, 7 x 7, 6 x 6, 5 x 5, …… 1 x 1 sizes

totaling 204

taking these squares also a rectangle. Total number of rectangles

= 1092
+ 204
1296

Q10. The squares of number

$40^2 = 1600$ $41^2 = 1681$

$42^2 = 1764$ $43^2 = 1849$

$44^2 = 1936$ As Mr. John died in year 1871, so he was either

42 years old in 1764 and died in 1871

or 43 years old in 1849 and died in 1871

The option of 42 years appears to be invalid

Hence he was 43 years old in 1871.

Hence born in year 1871
- 43
= 1828